WAHC 2024 – 12th Workshop on Encrypted Computing & Applied Homomorphic Cryptography

Faster Homomorphic Evaluation of Arbitrary Bivariate Integer Functions via Homomorphic Linear Transformation

October 14th, 2024

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Introduction

Fully Homomorphic Encryption (FHE)

- FHE: a cryptographic primitive that allows computation on encrypted data:
  - For plaintexts a, b and their encryptions $[\![a]\!], [\![b]\!]$, we can compute
    - $[\![a]\!] \oplus [\![b]\!] = [\![a + b]\!]$
    - $[\![a]\!] \otimes [\![b]\!] = [\![a \times b]\!]$

$[\![a]\!]$: an encryption of a

- In our work, we focus on the BFV scheme:
  - Suitable for integer-wise operations ($\oplus$ and $\otimes$).
  - Supports the packing method [SV14]
    - Which enables slot-wise computations for vectors :
      - $[\![(a_0, \ldots, a_{N-1})]\!] \oplus [\![(b_0, \ldots, b_{N-1})]\!] = [\![(a_0 + b_0, \ldots, a_{N-1} + b_{N-1})]\!]$
      - $[\![(a_0, \ldots, a_{N-1})]\!] \otimes [\![(b_0, \ldots, b_{N-1})]\!] = [\![(a_0 \times b_0, \ldots, a_{N-1} \times b_{N-1})]\!]$
  - Supports automorphism operations
    - Which allows special operations (e.g., rotations, total-sum, and more)
    - And relatively faster than homomorphic multiplication.
    - σ_j : jth automorphism

[SV14] N. P. Smart and F. Vercauteren. 2014. Fully Homomorphic SIMD Operations. Designs, Codes and Cryptography 71, 1 (2014), 57–81.

Evaluation of Non-linear functions for Integer-wise FHE

- For BFV (and BGV), evaluating non-linear functions, such as division, is a challenging task.
  - The computation time of non-linear functions is significantly larger than that of the basic arithmetic operations (such as addition or multiplication).

- Prior works:
  - Okada et al. [OCHK18] have proposed a method for evaluating arbitrary bivariate functions using BGV.
    - Efficient for small input domain (few bits)
    - Too impractical for relatively large input domains
      - e.g. estimated that 15 bits division takes 117.3 days…
  - Maeda et al. [MMNFT24] have proposed a method for evaluating arbitrary bivariate functions with relatively large bit lengths **instead of special plaintext encoding**, which is much faster than the method by Okada et al.

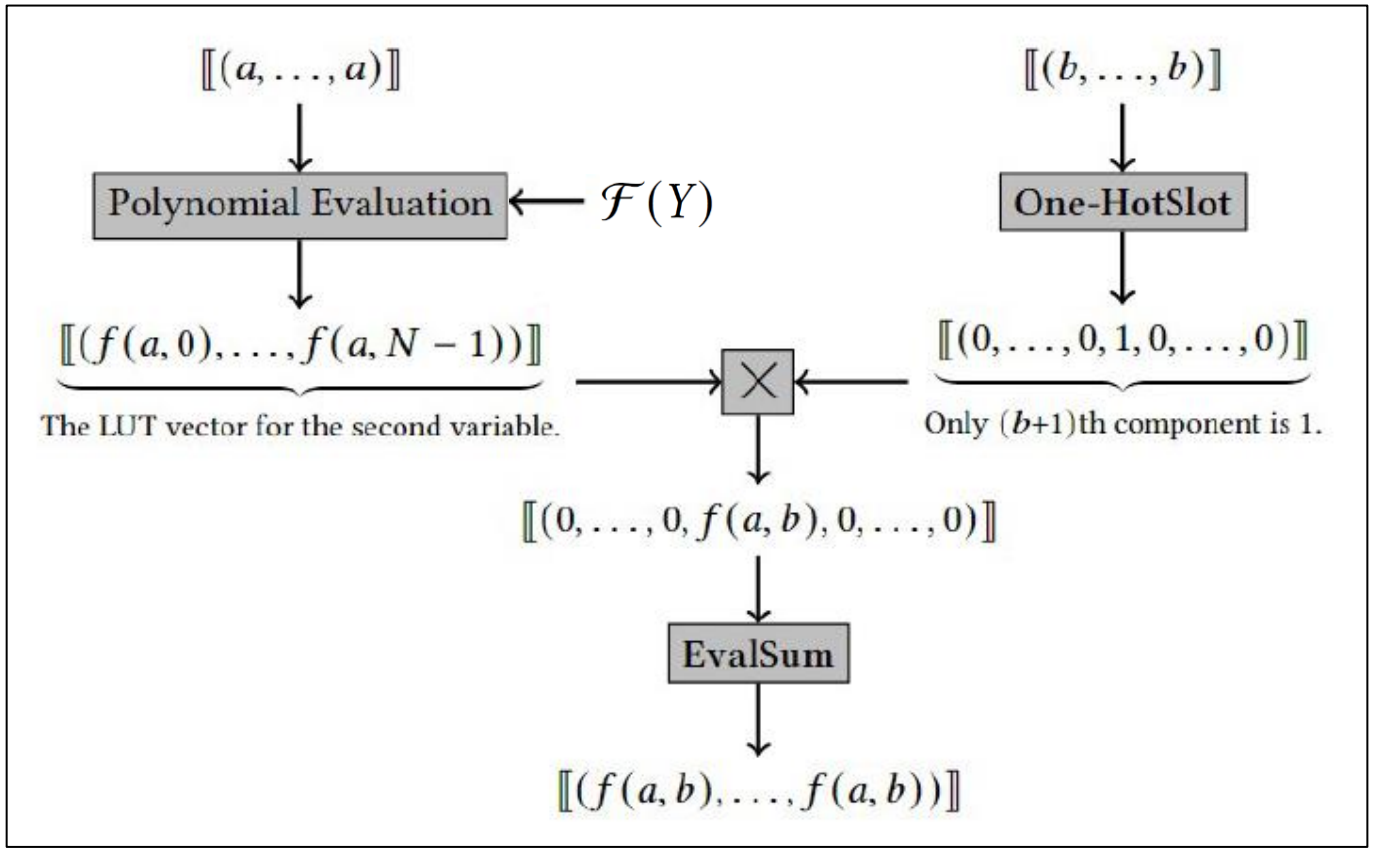
- Contribution:
  - We propose a **faster method** than the method by Maeda et al. with the same plaintext encoding.

[OCHK18] H. Okada, et al. ”Linear depth integer-wise homomorphic division.” WISTP2018, 2018
[MMNFT24] D. Maeda, K. Morimura, S. Narisada, K. Fukushima, and T. Nishide. 2024. Efficient Homomorphic Evaluation of Arbitrary Uni/Bivariate Integer Functions and Their Applications. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E107.A, 3 (2024), 234–247.

Prior Work by Maeda et al. [MMFFT24]

Prior Work: Evaluation of Arbitrary Bivariate Functions

- The method by Maeda et al. enables handling **relatively large input domain** in exchange for **filling all slots with a single value,** i.e., we use $[\![(a, \ldots, a)]\!]$ for a plaintext a.
- To maximize the number of slots:
  - We use a prime plaintext modulus t s.t. $t \equiv 1 \pmod{2N}$ for a ring dimension N (power of two).



Precomputation

- For a bivariate function $f : \{0,1 \ldots, N-1\} \times \{0,1 \ldots, N-1\} \to \mathbb{F}_t$, we can define a univariate function $f_y(x) = f(x, y)$ by fixing the second variable y.

- Since the range of $f_y(x)$ is included in a finite field $\mathbb{F}_t$, we can express $f_y(x)$ as a polynomial

$$f_y(x) = f_{0,y} + f_{1,y} \cdot x + \cdots + f_{N-1,y} \cdot x^{N-1} \in \mathbb{F}_t[x]$$

  for the first variable.

- For each $y = 0, 1, \ldots N-1$, precompute the coefficients of $f_y(x)$ and store each coefficient vector.

$$(f_{0,0}, \cdots, f_{i,0}, \cdots, f_{N-1,0})$$ Coefficients of $f_0(x)$

$$(f_{0,1}, \cdots, f_{i,1}, \cdots, f_{N-1,1})$$ Coefficients of $f_1(x)$

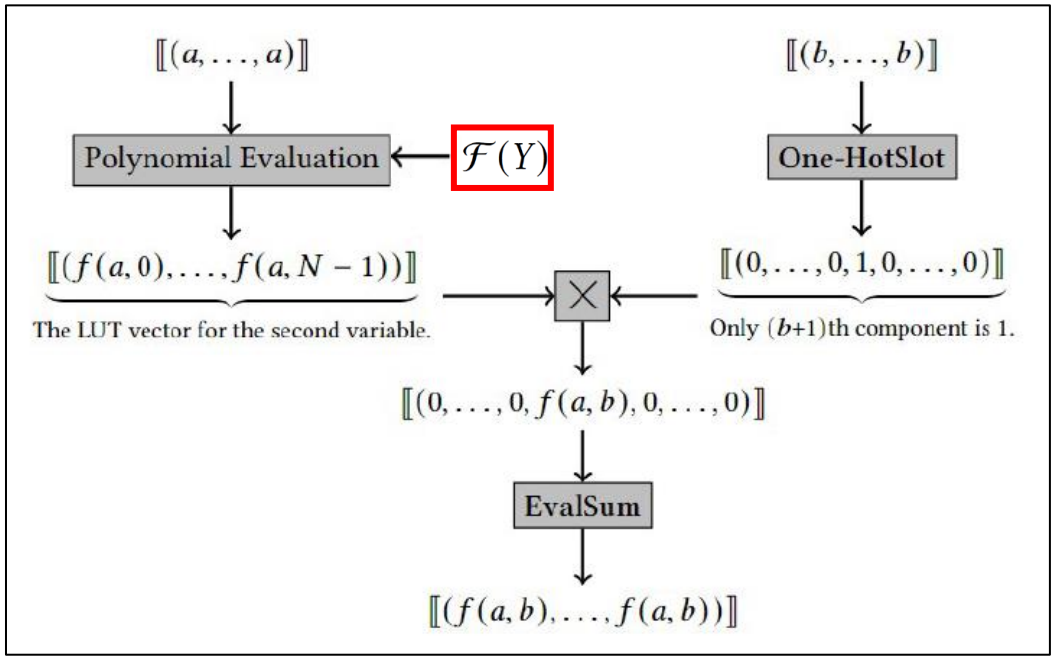
$$\vdots$$

$$(f_{0,N-1}, \cdots, f_{i,N-1}, \cdots, f_{N-1,N-1})$$ Coefficients of $f_{N-1}(x)$

- Then, we define

$$\mathcal{F}(z) := \sum_{i=0}^{N-1} [(f_{i,0}, f_{i,1}, \cdots, f_{i,N-1})] \cdot z^i \in R_t[z]$$

Packed vertically



Polynomial Evaluation for the First Variable

- By homomorphically substituting the ciphertext $[\![(a, \ldots, a)]\!]$ for the polynomial defined before

$$\mathcal{F}(z) = \sum_{i=0}^{N-1} \left[\left(f_{i,0}, f_{i,1}, \ldots, f_{i,N-1}\right)\right] \cdot z^i \in R_t[z],$$

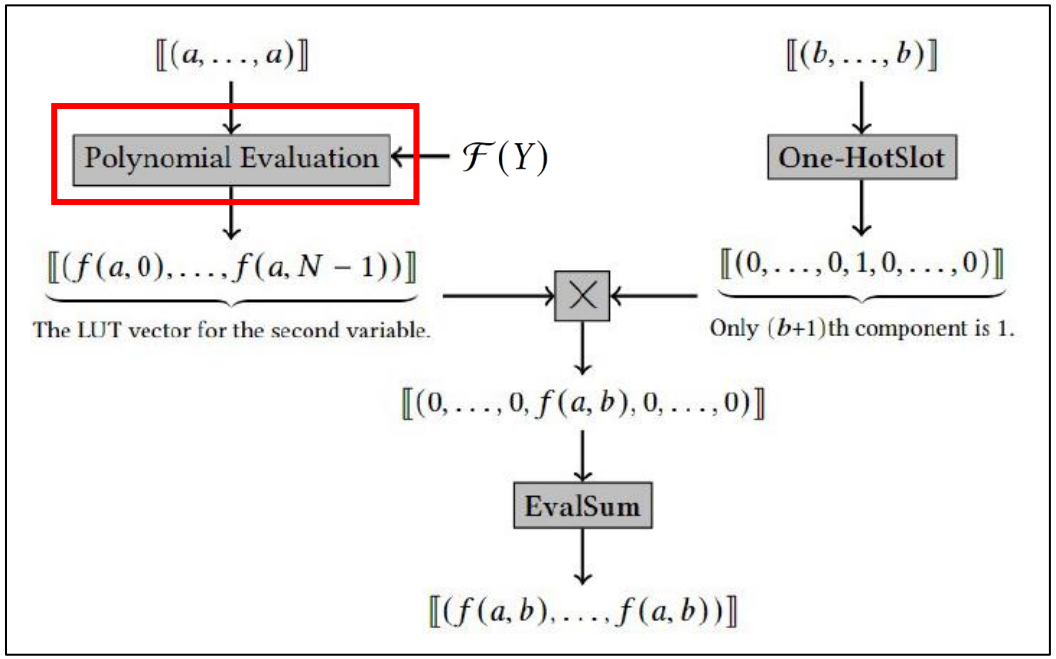
  we can obtain **a ciphertext of a LUT vector for second variables**, since

$$\mathcal{F}([\![(a, \ldots, a)]\!]) = \sum_{i=0}^{N-1} \left[\left(f_{i,0}, f_{i,1}, \ldots, f_{i,N-1}\right)\right] \cdot [\![(a^i, \ldots, a^i)]\!] = [\![(f_0(a), f_1(a), \ldots, f_{N-1}(a))]\!] = [\![(f(a,0), f(a,1) \ldots, f(a,N-1))]\!]$$

Evaluated $f_0(x), f_1(x), \ldots, f_{N-1}(x)$ in parallel.

- The above polynomial evaluation can be computed by $O(\log(N))$ depth.

- The total number of homomorphic multiplications is about $2\sqrt{N}$ via the Paterson–Stockmeyer method [PS73], but it is still too slow…



[PS72] M. Paterson and L. J. Stockmeyer. 1973. On the Number of Nonscalar Multiplications Necessary to Evaluate Polynomials. SIAM J. Comput. 2, 1 (1973), 60–66.

Conversion of the Second Variable to the One-Hot Vector

- By subtracting $[(0, 1, \ldots, N-1)]$ from the second ciphertext $[\![(b, b, \ldots, b)]\!]$

$$[\![(b, b, \ldots, b)]\!] - [(0, 1, \ldots, N-1)] = [\![(b, b-1, \ldots, 0, \ldots, b-N+1)]\!]$$

  obtain a ciphertext of a vector s.t. only $(b+1)$th component is zero and others are non-zero.

- The above ciphertext to the power t is

Thanks to Fermat little theorem

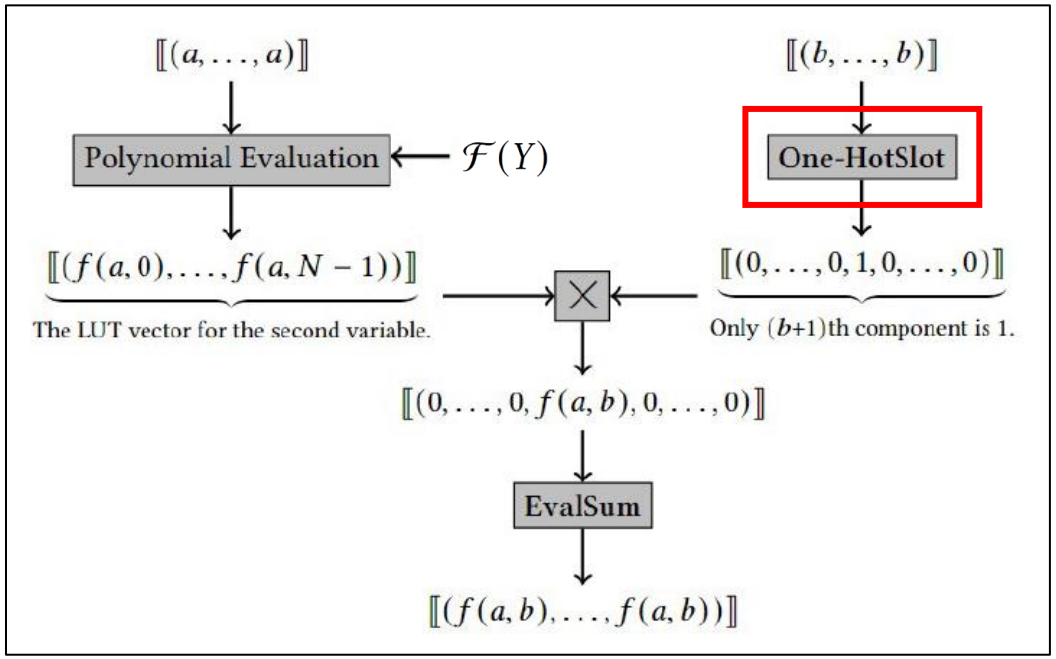
$$[\![(b, b-1, \ldots, 0, \ldots, b-N+1)]\!]^{t-1} = [\![(1, 1, \ldots, 0, \ldots, 1)]\!]$$

- By subtracting the above ciphertext from the packed plaintext $[1,1,\ldots,1]$ filled with 1, we obtain a ciphertext of a one-hot slot vector

$$[\![(0, \ldots, 1, \ldots, 0)]\!]$$

Only $(b+1)$th component is 1.

- This One-HotSlot procedure can be efficiently computed since it only requires about $\log(t)$ times homomorphic multiplications.



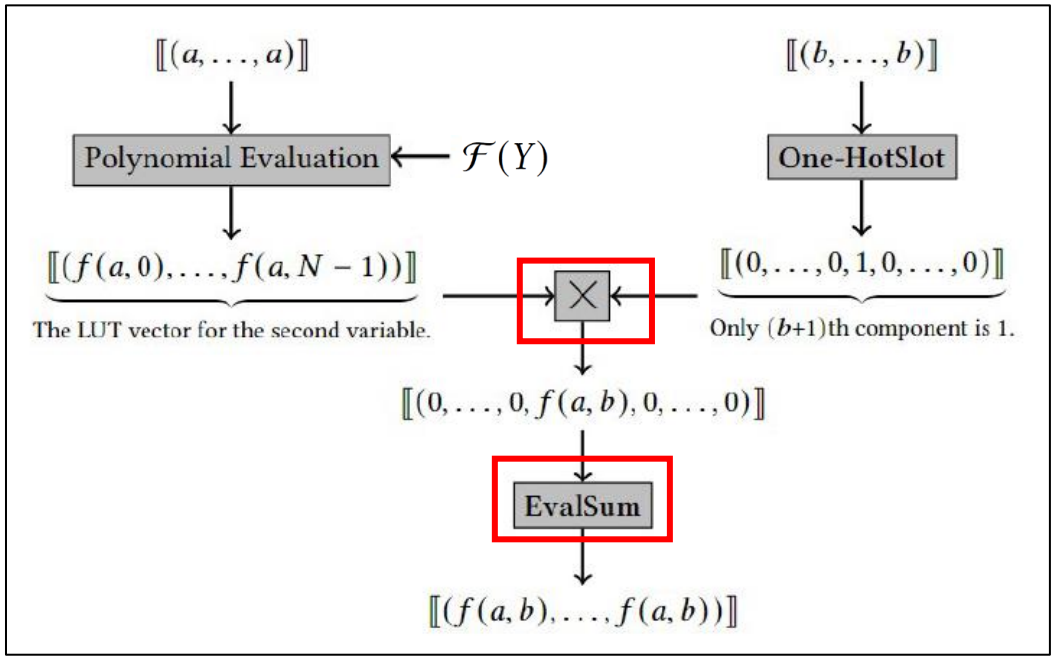
Extraction from the LUT vector

- By multiplying the ciphertext of the LUT vector and the ciphertext of the one-hot vector, we can extract $f(a,b)$

$$[\![(0,\dots,f(a,b),\dots,0)]\!] = [\![(f(a,0),\dots,f(a,N-1))]\!] \otimes [\![(0,\dots,1,\dots,0)]\!]$$

- After that, taking the summation of all slots by performing the EvalSum operation (which requires $\log(N)$ automorphisms), we finally obtain $[\![(f(a,b),\dots,f(a,b))]\!]$.

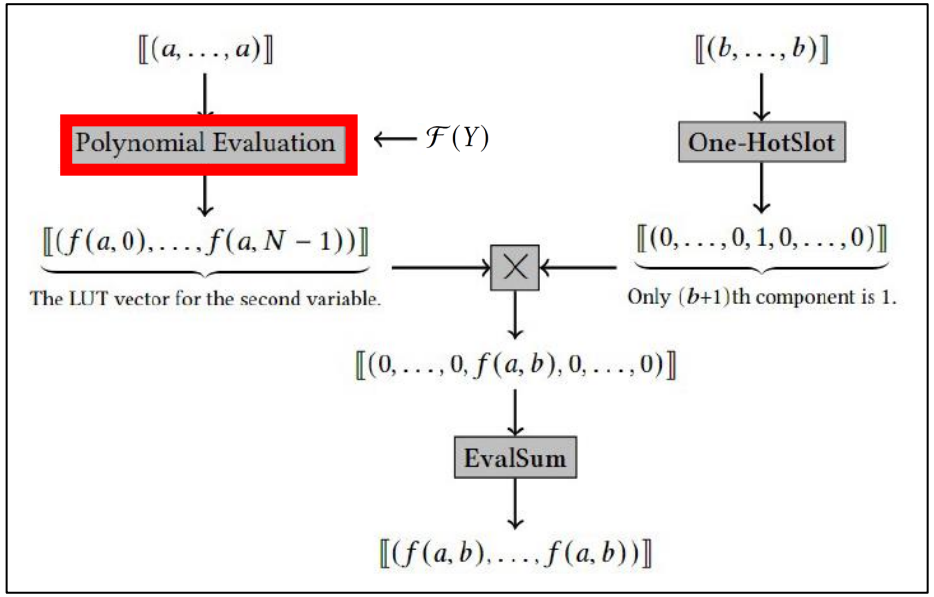
- This extraction step has few effects on the total time of the evaluation of a bivariate function since this step only requires
  - Single homomorphic multiplication
  - $\log(N)$ automorphisms
    - Automorphism is a relatively faster operation than homomorphic multiplication.



Our Proposal

Idea of Our Proposal

- In the prior work by Maeda et al., transforming a first variable ciphertext $[\![(a,\ldots,a)]\!]$ to a LUT vector $[\![(f(a,0),\ldots,f(a,N-1))]\!]$ is the most time-consuming part,
  since it requires **many homomorphic multiplications**
  during evaluating the predefined polynomial.



- Our idea is to use a lookup table as a **matrix:**

$$F = \begin{pmatrix} f(0,0) & \cdots & f(0,N-1) \\ f(1,0) & \cdots & f(1,N-1) \\ \vdots & \ddots & \vdots \\ f(N-1,0) & \cdots & f(N-1,N-1) \end{pmatrix}$$

- We can multiply the matrix F to a packed ciphertext via **homomorphic linear transformation.**

Homomorphic Linear Transformation

- Homomorphic linear transformation [HS18] is a technique for multiplying a packed ciphertext by a matrix, i.e., for a packed ciphertext $[\![\vec{a}]\!] = [\![(a_0, \ldots, a_{N-1})]\!]$ and a marix M (square matrix of order N), we can compute

$$[\![\vec{a} \cdot M]\!] = [\![\vec{a}]\!] \cdot M$$

  via homomorphic linear transformation.

- When a square matrix of order N does not need to be encrypted, homomorphic linear transformation canperform by about $\sqrt{2N}$ times automorphisms (via the baby-step giant-step technique).

- Then, homomorphic linear transformation for a square matrix of order N must be faster than the evaluation of a degree-$(N-1)$ polynomial (which requires about $2\sqrt{N}$ times homomorphic multiplications)

Algorithm 5 HLT

Input: $\boldsymbol{c} = [\![\boldsymbol{a}]\!], M \in \mathbb{F}_t^{N \times N}$
Output: $[\![\boldsymbol{a} \cdot M]\!]$

1: (Precomputing) : For a matrix M, compute $\overline{\boldsymbol{m}}_{5^{\alpha+A\cdot\beta}}, \underline{\boldsymbol{m}}_{(-1)\cdot 5^{\alpha+A\cdot\beta}}$ $(0 \le \alpha \le A-1, 0 \le \beta \le B-1)$.
2: $\boldsymbol{d}_{\text{digits}} \leftarrow \mathbf{Ext}_{Q\to PQ}(\mathcal{D}(c_1))$
3: Put $\boldsymbol{d}_{\text{digits}}$ in NTT format
4: $\boldsymbol{c}_L \leftarrow [0], \boldsymbol{c}_R \leftarrow [0]$
5: **for** $\beta = 0$ to $B-1$ **do**
6: $\quad \boldsymbol{c}_{L_{\text{BS}}} \leftarrow \boldsymbol{c} \cdot [\overline{\boldsymbol{m}}_{5^{A\cdot\beta}}], \boldsymbol{c}_{R_{\text{BS}}} \leftarrow \boldsymbol{c} \cdot [\underline{\boldsymbol{m}}_{-5^{A\cdot\beta}}]$ // $\alpha = 0$
7: $\quad$ **for** $\alpha = 1$ to $A-1$ **do**
8: $\quad\quad \tilde{\boldsymbol{c}}_1 \leftarrow [\boldsymbol{d}_{\text{digits}} \cdot \mathbf{ksk}_{s\to\sigma_{5^\alpha}^{-1}(s)}]_{PQ}$
9: $\quad\quad \sigma_{5^\alpha}(\boldsymbol{c}) \leftarrow (\sigma_{5^\alpha}(c_0) + \lfloor \sigma_{5^\alpha}(\tilde{c}_{1,0})/P \rceil, \lfloor \sigma_{5^\alpha}(\tilde{c}_{1,1})/P \rceil)$
10: $\quad\quad \boldsymbol{c}_{L_{\text{BS}}} \leftarrow \boldsymbol{c}_{L_{\text{BS}}} \oplus \sigma_{5^\alpha}(\boldsymbol{c}) \cdot [\overline{\boldsymbol{m}}_{5^{\alpha+A\cdot\beta}}]$
11: $\quad\quad \boldsymbol{c}_{R_{\text{BS}}} \leftarrow \boldsymbol{c}_{R_{\text{BS}}} \oplus \sigma_{5^\alpha}(\boldsymbol{c}) \cdot [\underline{\boldsymbol{m}}_{-5^{\alpha+A\cdot\beta}}]$
12: $\quad \boldsymbol{c}_L \leftarrow \boldsymbol{c}_L \oplus \sigma_{5^{A\cdot\beta}}(\boldsymbol{c}_{L_{\text{BS}}})$
13: $\quad \boldsymbol{c}_R \leftarrow \boldsymbol{c}_R \oplus \sigma_{5^{A\cdot\beta}}(\boldsymbol{c}_{R_{\text{BS}}})$
14: $\tilde{\boldsymbol{c}} \leftarrow \boldsymbol{c}_L \oplus \sigma_{-1}(\boldsymbol{c}_R)$
15: **return** $\tilde{\boldsymbol{c}} (= [\![\boldsymbol{a} \cdot M]\!])$

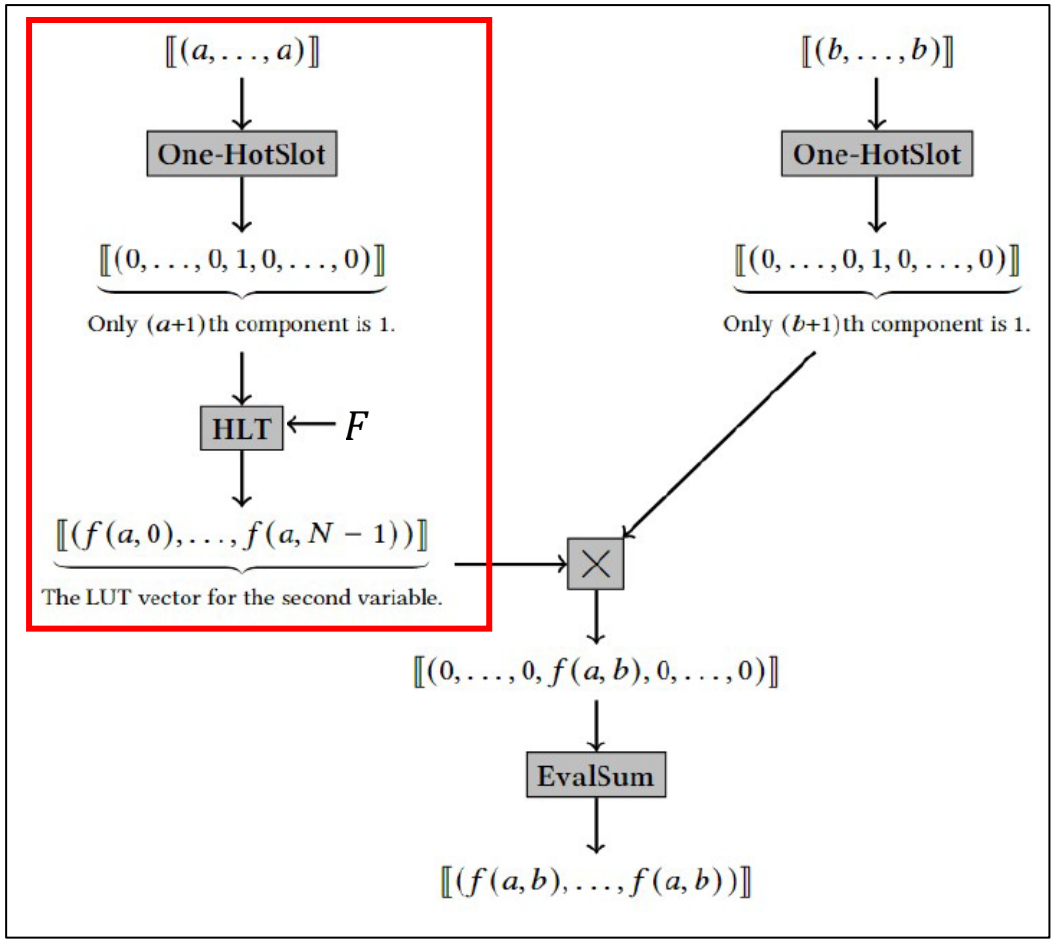
[HS18] S. Halevi and V. Shoup. 2018. Faster Homomorphic Linear Transformations in HElib. In Advances in Cryptology – CRYPTO 2018 (Lecture Notes in Computer Science). Springer, 93–120.

Our Proposal

- We can efficiently translate $[\![(a, \ldots, a)]\!]$ to a ciphertext of a one-hot vector $[\![(0, \ldots, 1, \ldots, 0)]\!]$ for the first variable.
- The one-hot vector for the first variable a can be used to extract the $(a+1)$th row of the matrix F by multiplying it homomorphically.

$$\underbrace{(0, \ldots, 0, 1, 0, \ldots, 0)}_{\text{Only } (a+1)\text{th component is 1.}} \cdot \begin{pmatrix} f(0,0) & \cdots & f(0,N-1) \\ \vdots & \ddots & \vdots \\ f(a,0) & \cdots & f(a,N-1) \\ \vdots & \ddots & \vdots \\ f(N-1,0) & \cdots & f(N-1,N-1) \end{pmatrix}$$

- Eliminate the polynomial evaluation process, which is the most time-consuming part.



Comparison

Complexity

- Maeda et al.'s method requires
  - $O(\sqrt{N})$ homomorphic multiplications (for the polynomial evaluation and One-HotSlot)
  - $O(\log(N))$ automorphisms (for EvalSum)

- Our proposal requires:
  - $O(\log(N))$ homomorphic multiplications (for One-HotSlot × 2)
  - $O(\sqrt{N})$ automorphisms (for homomorphic linear transformation and EvalSum)

- This leads to an efficiency gap between these methods.

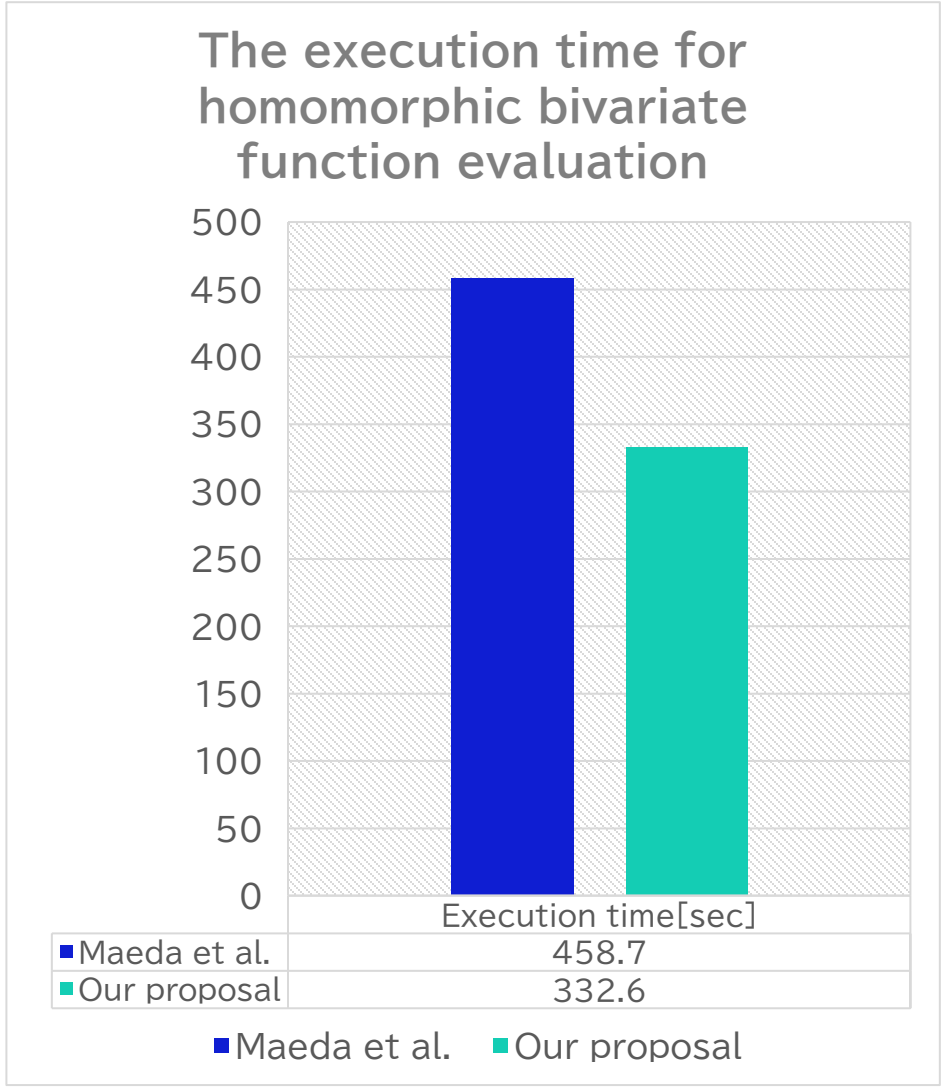
- As a drawback, our proposal requires more key-switching keys : $O(\log(N)) \rightarrow O(\sqrt{N})$

Method	#-Homomorphic multiplications	#-Automorphisms
Maeda et al.	$O(\sqrt{N})$	$O(\log(N))$
Ours	$O(\log(N))$	$O(\sqrt{N})$

[MMNFT24] D. Maeda, K. Morimura, S. Narisada, K. Fukushima, and T. Nishide. 2024. Efficient Homomorphic Evaluation of Arbitrary Uni/Bivariate Integer Functions and Their Applications. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E107.A, 3 (2024), 234–247.

Implementation Result

- Experimental Environment:
  - CPU: Intel Xeon Silver 4114@2.20GHz
  - Memories: 96GB
  - Library: OpenFHE
  - Scheme: BFVrns
  - Security Parameter: HEStd_128_classic
- The input domain is $\{0,1,\dots,N-1\} \times \{0,1,\dots,N-1\}$ for $N = 2^{15}$.
- We choose the division function $f(x,y) = \lfloor x/y \rfloor$ as a representative non-linear bivariate function.
- Our proposed method reduces the execution time to **72.5%**, which decreases the total time by **126.1 seconds**.



Conclusions

- We proposed a method for evaluating arbitrary bivariate functions, which is faster than the method proposed by Maeda et al.
  - Less homomorphic multiplications ($O(\sqrt{N}) \to O(\log(N))$), more automorphisms ($O(\log(N)) \to O(\sqrt{N})$)
    - It decreases the computation time
    - But it requires more key-switching keys… ($O(\log(N)) \to O(\sqrt{N})$)
  - Through the implementation by OpenFHE, we confirm that our method achieved a speed-up compared to the previous method.

- Future work:
  - Extend input domain from N to t ($> 2N$) by using multiple ciphertexts similar to [MMNFT24], and implement it.

[MMNFT24] D. Maeda, K. Morimura, S. Narisada, K. Fukushima, and T. Nishide. 2024. Efficient Homomorphic Evaluation of Arbitrary Uni/Bivariate Integer Functions and Their Applications. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E107.A, 3 (2024), 234–247.

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Estimated Memory Usage

- For the parameters in our experiment, the size of each key switching key ksk is estimated 20.6 MB.
  - The method by Maeda et al. requires 289.0 MB
  - The proposed method requires 6874.4 MB (≒ 6.9 GB)
    - Almost 24× memory usage.

[MMNFT24] D. Maeda, K. Morimura, S. Narisada, K. Fukushima, and T. Nishide. 2024. Efficient Homomorphic Evaluation of Arbitrary Uni/Bivariate Integer Functions and Their Applications. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E107.A, 3 (2024), 234–247.